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Prokhorov Radius of a Neighborhood of Zero Described by Three Moment Constraints

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Abstract. We determine the Prokhorov radius of the family of distributions surrounding the Dirac measure at zero whose first, second and fourth moments are bounded by given numbers. This provides the precise relation between the rates of weak convergence to zero and the rate of vanishing of the respective moments.

Key words: Moment conditions, Prokhorov metric, Rate of convergence, Tchebycheff system, Weak convergence

1. Introduction and main result

We start with recalling the notion of Prokhorov distance of two probability measures μ, ν, which is generally defined on a Polish space with a metric d. This is given by

$$\pi(\mu,\nu) = \inf\{r>0 : \mu(A) \leqslant \nu(A^r)+r,\ \nu(A) \leqslant \mu(A^r)+r,\ \text{for every closed subset } A\},$$

where $A^r = \{x : d(x,A) < r\}$. Note that in the case of standard real space with the Euclidean metric, the Prokhorov distance of a probability measure μ to the degenerate one δ_0 concentrated at 0 can be written as

$$\pi(\mu,\delta_0) = \inf\{r>0 : \mu(I_r) \geqslant 1-r\}. \tag{1}$$

Here and later on $I_r = [-r, r]$, and I_r^c stands for its complement.

For a given teiple of positive reals $\mathcal{E} = (\epsilon_1, \epsilon_2, \epsilon_4)$, we consider the family $\mathcal{M}(\mathcal{E})$ of probability measures on the real line such that

$$\mathcal{M}(\mathcal{E}) = \left\{\mu : \left|\int t^i \, d\mu\right| < \epsilon_i,\ i = 1, 2, 4\right\}. \tag{2}$$

Theorem 1 provides the precise evaluation of the Prokhorov radius

$$D(\mathcal{E}) = \sup_{\mu\in\mathcal{M}(\mathcal{E})} \pi(\mu,\delta_0)$$

for family of measures (2).

THEOREM 1. *We have*

$$D(\epsilon_1, \epsilon_2, \epsilon_4) = \min\{\epsilon_2^{1/3}, \epsilon_4^{1/5}\}. \tag{3}$$

This is a refinement of a result in Anastassiou (1992) where the Prokhorov radius $D(\epsilon_1, \epsilon_2) = \epsilon_2^{1/3}$ of the family with constraints on two first moments was established. The problems of determining the Levy and Kantorovich radii under two moment conditions were considered in Anastassiou (1987) and Anastassiou and Rachev (1992), respectively. Anastassiou and Rychlik (1999) studied the Prokhorov radius of measures supported on the positive halfaxis which satisfy conditions on the first three moments. Since the Prokhorov metric induces the topology of weak convergence, formula (3) describes the exact rate of weak convergence of measures from $\mathcal{M}(\mathcal{E})$ satisfying the three moment constraints to the Dirac one at zero.

Though our question is stated in an abstract way, it stems straightforwardly from applied probability problems in which rates of convergence of random error of a consistent statistical estimate vanishes, then zero is the most natural limiting point. Convergence in probability is implied by that of the first two moments. Adding the fourth one, which has a meaningful interpretation in statistics, allows us to obtain refined evaluations. These three moments have natural estimates, and so one can easily control their variability. Moreover, the respective power functions form a Tchebycheff system. Convergence of integrals for elements of such systems implies and provides estimates for integrals of general continuous functions. The latter convergence is described by the weak topology, and our solution gives a quantitative estimate of uniform weak convergence (expressed in terms of equivalent Prokhorov metric topology) for a large natural class of measures determined by moment conditions.

Formula (3) is determined by means of a geometric moment theoretical method of Kemperman (1968) that will be used in Section 2 for calculating

$$L_r(M) = \inf_{\mu \in \mathcal{M}(M)} \mu(I_r) \tag{4}$$

with given $M = (m_1, m_2, m_4)$ and

$$\mathcal{M}(M) = \{\mu : \int t^i \, d\mu = m_i, i = 1, 2, 4\}$$

for all possible m_1, m_2, m_4, and $r > 0$. In Section 3 we prove our main result; having determined (4) for various M, we first evaluate respective infima over the boxes in the moment space

$$L_r(\mathcal{E}) = \inf\{L_r(M) : |m_i| \leqslant \varepsilon_i, i = 1, 2, 4\} \tag{5}$$

for every fixed r, and then, leting r vary, we determine

$$D(\mathcal{E}) = \inf\{r > 0 : L_r(\mathcal{E}) \geqslant 1 - r\}. \tag{6}$$

In the short last section we sketch possible directions for a further research.

2. Auxiliary moment problem

Fixing $r > 0$, we now confine ourselves on solving moment problem (4). This is well stated iff

$$M \in \mathcal{W} = \{(m_1, m_2, m_4) : m_1 \in \mathcal{R}, m_2 \geqslant m_1^2, m_4 \geqslant m_2^2\}.$$

Note that $\mathcal{W} = \text{conv}\mathcal{T} = \text{conv}\{T = (t, t^2, t^4) : t \in \mathcal{R}\}$, the convex hull of the graph of function $\mathcal{R} \ni t \mapsto (t, t^2, t^4)$. Geometrically, $\mathcal{W}$ is a set unbounded above whose bottom $\underline{\mathcal{W}}$ is a membrane spanned by $\mathcal{T}$. The membrane can be represented as $\underline{\mathcal{W}} = \cup_{t\geqslant 0}\overline{T_-T_+}$, where $T_- = (-t, t^2, t^4)$, $T_+ = (t, t^2, t^4)$, and $\overline{AB}$ denotes the line segment with end-points A and B. The side surface consists of vertical halflines $T^\uparrow$ running upwards from the points $T \in \mathcal{T}$. Consider the following surfaces in $\mathcal{W}$:

$\Delta\mathbf{0}R_+R_-$ — the triangle with vertices $\mathbf{0} = (0, 0, 0)$, $R_+ = (r, r^2, r^4)$ and $R_- = (-r, r^2, r^4)$,

$\text{mem}(R_+, \widehat{\mathbf{0}R}_+) = \cup_{0\leqslant t\leqslant r}\overline{TR_+}$, and $\text{mem}(R_-, \widehat{\mathbf{0}R}_-) = \cup_{-r\leqslant t\leqslant 0}\overline{TR_-}$ — the membranes connecting R_+ and R_- with the points of the curves $\widehat{\mathbf{0}R}_+ = \{(t, t^2, t^4);\ 0 \leqslant t \leqslant r\}$ and $\widehat{\mathbf{0}R}_- = \{(-t, t^2, t^4) : 0 \leqslant t \leqslant r\}$, respectively,

$\overline{R_-R_+}^\uparrow$, $\overline{\mathbf{0}R_+}^\uparrow$, and $\overline{\mathbf{0}R_-}^\uparrow$ — the infinite bands above the line segments $\overline{R_-R_+}$, $\overline{\mathbf{0}R_+}$ and $\overline{\mathbf{0}R_-}$, respectively.

They partition $\mathcal{W}$ into five closed subsets with nonoverlapping interiors:

$\mathcal{W}_1$ – the set of points situated on and above $\Delta\mathbf{0}R_+R_-$,

$\mathcal{W}_2$ – the moment points on and above $\text{mem}(R_+, \widehat{\mathbf{0}R}_+)$,

$\mathcal{W}_3$ – the points on and above $\text{mem}(R_-, \widehat{\mathbf{0}R}_-)$,

$\mathcal{W}_4$ – the points between $\Delta\mathbf{0}R_+R_-$, $\text{mem}(R_+, \widehat{\mathbf{0}R}_+)$, $\text{mem}(R_-, \widehat{\mathbf{0}R}_-)$, and $\underline{\mathcal{W}}^{I_t} = \cup_{0\leqslant at\leqslant r}\overline{T_-T_+}$, the last surface being a part of the bottom of the moment space,

$\mathcal{W}_5$ – the moment points lying on and above $\mathcal{W}^{I_r^c} = \cup_{t\geqslant r}\overline{T_-T_+}$.

The solution to (4) is expressed by different formulae for the elements of the above partition.

THEOREM 2. *The solution to (4) is given by*

$$L_r(M) = \begin{cases} 1 - m_2/r^2, & \text{if } M \in \mathcal{W}_1, \\ (r - |m_1|)^2/(r^2 - 2|m_1|r + m_2), & \text{if } M \in \mathcal{W}_2 \cup \mathcal{W}_3, \\ (r^2 - m_2)^2/(r^4 - 2m_2r^2 + m_4), & \text{if } M \in \mathcal{W}_4, \\ 0, & \text{if } M \in \mathcal{W}_5. \end{cases} \tag{7}$$

One can easily verify that the formulae for neighboring regions coincide on their common borders. In particular, this implies continuity of L_r.

Proof of Theorem 2. First notice that $\mathcal{W}_5$ is the closure of the convex hull of $\mathcal{T}(I_r^c) = \{(t, t^2, t^4) : |t| > r\}$. The inner elements of $\mathcal{W}_5$ are the moment points for measures supported on I_r^c and therefore $L_r(M) = 0$ for all $M \in \mathcal{W}_5$.

The other formulae in (7) will be determined by means of the optimal ratio method due to Kemperman (1968) that allows us to find sharp lower and upper bounds for probability measures of a given set (here: the lower one for those of I_r) under the conditions that the integrals of some given functions with respect to the measures take on assumed values (here: $\int t^i \, d\mu = m_i, i = 1, 2, 4$). The method can be used under mild assumptions about the structure of probability space and functions appearing in the moment conditions (cf. Kemperman, 1968, Section 5). These are satisfied in the case we consider and therefore we merely present a version adapted to our problem instead of the general description. Given a boundary point W of $\mathcal{W}$, $W \notin \mathcal{W}_5$, we take a hyperplane $\mathcal{H}$ supporting $\mathcal{W}$ at W, and another one $\mathcal{H}'$ supporting $\mathcal{W}_5$ that is the closest one parallel to $\mathcal{H}$. Then for every moment point M in the closure of $\mathrm{conv}(\mathcal{W} \cap \mathcal{H}) \cup (\mathcal{W}_5 \cap \mathcal{H}')$, we have

$$L_r(M) = \frac{d(M, \mathcal{H}')}{d(\mathcal{H}, \mathcal{H}')}, \tag{8}$$

where the numerator and denominator in (8) denote the distances from the moment point M and hyperplane $\mathcal{H}$ to $\mathcal{H}'$, respectively.

We shall therefore take into account the hyperplanes $\mathcal{H}$ supporting points $W \in \cup_{|t|<r} T^{\uparrow} \cup \underline{\mathcal{W}}^{I_r}$. First consider the vertical plane $\mathcal{H} : m_2 = 0$ that supports $\mathcal{W}$ at all points of $\mathbf{0}^{\uparrow}$. Then $\mathcal{H}' : m_2 - r^2 = 0$ is the closest and parallel to $\mathcal{H}$ plane that supports $\mathcal{W}_5$. Since $\mathcal{H} \cap \mathcal{W} = \mathbf{0}^{\uparrow}$, and $\mathcal{H}' \cap \mathcal{W}_5 = \overline{R_- R_+}^{\uparrow}$, then for every $M \in \mathrm{conv}\mathbf{0}^{\uparrow} \cup \overline{R_- R_+}^{\uparrow} = \mathcal{W}_1$, we apply (8) to obtain $L_r(M) = 1 - m_2/r^2$.

Consider now a side hyperplane $\mathcal{H}$ such that $\mathcal{H} \cap \mathcal{W} = T^{\uparrow}$ for some $0 < t < r$, described by the formula $\mathcal{H} : m_2 - 2tm_1 + t^2 = 0$. We can easily see that $\mathcal{H}'$ is the plane parallel to $\mathcal{H}$ that supports $\mathcal{H}_5$ along $R_+^{\uparrow}$. This can be written as $\mathcal{H}' : m_2 - 2tm_1 + 2tr - r^2 = 0$. Applying the standard formula

$$d(Y, \mathcal{A}) = \left| \sum_{i=1}^{n} a_i y_i + b \right| \Big/ \left(\sum_{i=1}^{n} a_i^2 \right)^{1/2}$$

measuring the Euclidean distance between a point $Y = (y_1, \ldots, y_n)$ and a hyperplane $\mathcal{A} : \sum_{i=1}^{n} a_i x_i + b = 0$ in $\mathcal{R}^n$, we get

$$d(M, \mathcal{H}') = |m_2 - 2m_1 t + 2tr - r^2|/(1 + 4t^2)^{1/2},$$
$$d(\mathcal{H}, \mathcal{H}') = d(\mathcal{H}, R_+) = (r - t)^2/(1 + 4t^2)^{1/2},$$

and, in consequence,

$$L_r(M) = \frac{|m_2 - 2m_1 t + 2tr - r^2|}{(r - t)^2} \tag{9}$$

for all $m \in \mathrm{conv} T_+^\uparrow \cup R_+^\uparrow = \overline{T_+R_+}^\uparrow$. Representing M as a point of the plane containing $\overline{T_+R_+}^\uparrow$, we obtain $m_2 = (r+t)(m_1 - r) + r^2$, which enables us to express t in terms of m_1 and m_2 as

$$t = \frac{(m_1 r - m_2)}{(r - m_1)}.$$

This substituted into (9) yields

$$L_r(M) = \frac{(r - m_1)^2}{r^2 - 2m_1 r + m_2}. \tag{10}$$

Note that this holds for all $M \in \mathcal{W}_2 = U_{0 \leqslant t \leqslant r} \overline{T_+R_+}^\uparrow$.

The respective formula for $M \in \mathcal{W}_3$ is obtained by replacing m_1 by $-m_1$ in (10). This is justified by the fact that the arguments of the optimal ratio method are purely geometric, and both $\mathcal{W}$ and $\mathcal{W}_5$ are symmetric about the plane $m_1 = 0$.

Consider now a plane $\mathcal{H}$ that touches the bottom side of $\mathcal{W}$ along $\overline{T_-T_+}$ for some $0 < t < r$. This is defined by the formula $\mathcal{H} : m_4 - 2t^2 m_2 + t^4 = 0$. Then $\mathcal{H}' : m_4 - 2t^2 m_2 - r^4 + 2t^2 r^2 = 0$ is the closest parallel hyperplane to $\mathcal{H}$ that supports $\mathcal{W}_5$ along $\overline{R_-R_+}$. Arguments similar to those applied in the analysis of the side hyperplanes yield

$$d(M, \mathcal{H}') = |m_4 - 2m_2 t^2 + 2t^2 r^2 - r^4| / (1 + 4t^4)^{1/2}, \tag{11}$$

$$d(\mathcal{H}, \mathcal{H}') = (r^2 - t^2)^2 / (1 + 4t^4)^{1/2}, \tag{12}$$

where

$$t^2 = \frac{m_2 r^2 - m_4}{r^2 - m_2} \tag{13}$$

is determined from the equation $m_4 = (r^2 + t^2)(m_2 - r^2) + r^4$, defining the plane that contains both $\overline{T_-T_+}$ and $\overline{R_-R_+}$. Dividing (11) by (12) and substituting (13) for t^2 gives the penultimate formula in (7). Observe that this is valid for the moment points of the trapezoids $\mathrm{conv}\overline{T_-T_+} \cup \overline{R_-R_+}$, $0 \leqslant t \leqslant r$, whose union forms $\mathcal{W}_4$. This ends the proof of Theorem 2. □

3. Proof of Theorem 1

We first verify that fixing m_2 and m_4 we minimize $L_r(m_1, m_2, m_4)$ at $m_1 = 0$. Note that $L_r(M)$ for $M \in \mathcal{W}_1 \cup \mathcal{W}_4 \cup \mathcal{W}_5$ does not depend on the value of m_1, and $(m_1, m_2, m_4) \in \mathcal{W}_i$ implies $(0, m_2, m_4) \in \mathcal{W}_i$, $i = 1, 4, 5$. Differentiating the second formula of (7) with respect to $|m_1|$, we obtain

$$\frac{\partial L_r(M)}{\partial |m_1|} = \frac{2(r - |m_1|)(|m_1| r - m_2)}{(r^2 - 2|m_1| r + m_2)^2},$$

which is nonnegative for $M \in \mathcal{W}_2 \cup \mathcal{W}_3$, because $|m_1| \leqslant r$ and $m_2 \leqslant |m_1|r$ there. Therefore we decrease $L_r(M)$ moving $M \in \mathcal{W}_2 \cup \mathcal{W}_3$ perpendicularly towards the plane $m_1 = 0$ until we reach the border. Then we can move further entering either $\mathcal{W}_1$ or $\mathcal{W}_4$ that would not result in change of $L_r(M)$ until we finally arrive at $(0, m_2, m_4)$.

Evaluating (5) we can therefore concentrate on the moment points from the rectangular

$$\mathcal{R}_0(\{(0, m_2, m_4) : m_i \leqslant \epsilon_i, i = 2, 4\}. \tag{14}$$

The points of $\mathcal{R}_0 \cap \mathcal{W}$ may generally belong to any of $\mathcal{W}_1$, $\mathcal{W}_4$ and $\mathcal{W}_5$. However, if some $M \in \mathcal{R}_0 \cap \mathcal{W}_5$, which is possible when $\epsilon_2 \geqslant r^2$ and $\epsilon_4 \geqslant r^4$, then $L_r(M) = L_r(\mathcal{E}) = 0$, which is useless in determining (6). Otherwise the moment points of $\mathcal{R}_0$ belong to either $\mathcal{W}_1$ or $\mathcal{W}_4$. In the former case L_r is evidently decreasing in m_2 and does not depend on M_4 see (7)). In the latter, L_r is decreasing in m_4, and increasing in m_2, because $m_2 \leqslant r^2$, $m_4 \leqslant m_2 r^2$, and so

$$\frac{\partial L_r(M)}{\partial m_2} = \frac{2(r^2 - m_2)(m_2 r^2 - m_4)}{(r^4 - 2m_2 r^2 + m_4)^2} \geqslant 0$$

for $M \in \mathcal{R}_0 \cap \mathcal{W}_4$.

We now claim that L_r is minimized on (14) at $E = (0, \epsilon, \epsilon r^2)$ with $\epsilon = \min\{\epsilon_2 \epsilon_4 / r^2\}$ so that

$$L_r(\mathcal{E}) = L_r(E) = 1 - \epsilon^2/r^2. \tag{15}$$

The latter equation follows from the fact that the $E \in \mathcal{W}_1$. We prove the former using the following arguments. First observe that if $\epsilon_4 > \epsilon_2 r^2$, we can exclude from considerations all points situated above $\overline{E_0 E}$ for $E_0 = (0, 0, \epsilon r^2)$. Indeed, any point $M = (0, m_2, m_4)$ of this area can be replaced by $M' = (0, m_2, \epsilon r^2) \in \overline{E_0 E}$ so that $L_r(M') = L_r(M)$. Then we exclude all points of $\mathcal{R}_0$ below $\overline{\mathbf{0}E}$, which belong to $\mathcal{W}_4$. Keeping m_4 fixed and decreasing m_2 until we reach $\overline{\mathbf{0}E}$, we actually decrease L_r. What still remains to analyze is $\Delta \mathbf{0} E_0 E$ to the level $\overline{E_0 E}$, and finally move them right to E which results in decreasing L_r.

Now we are only left with the task of determining (6) which, by (15), consists in solving the equation $1 - \epsilon^2/r^2 = 1 - r$, or, equivalently,

$$\min\{r^2 \epsilon_2, \epsilon_4\} = r^5. \tag{16}$$

If $\epsilon_4^{1/5} \leqslant \epsilon_2^{1/3}$ then the graphs of both sides of (16) cross each other at level ϵ_4, and the solution is $\epsilon_4^{1/5}$. Otherwise they meet below ϵ_4 for $r = \epsilon_2^{1/3}$. These conclusions establish the assertion of Theorem 1. □

4. Concluding remarks

A natural extension of the above problem consists in analyzing distributions tending to points different from zero. However, by reference to Anastassiou and Rychlik

(1999), in this case one can hardly expect obtaining final results in form of nice explicit formulae. Another question of interest is the Prokhorov radius described by other moments. Also, one can replace Tchebycheff systems of specific powers by elements of general families of functions, e.g. convex and symmetric ones. A next step of the project is determining radii of classes described by moment conditions in other metrics which induce the topology of weak convergence (see Anastassiou (1987), Anastassiou and Rachev (1992)). Comparing rates of convergence of radii of given classes of measures in various metrics would shed some new light on mutual relations of the metrics.

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